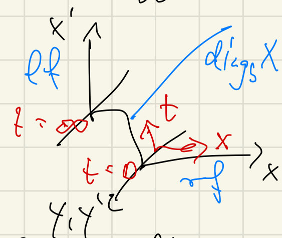
2021-01-20

Recall: $P = \sum_{k+|\alpha|\le m} a_{k\alpha}(x,y)(xD_x)^k D_y^\alpha$
b-diff op.

has int. kernel $\tilde{\beta}^*\left(\delta(1-t)\,\delta(y-y')\right)\sqrt{\frac{dx}{x}\frac{dt}{t}}$

Dirac section on X^2_b w.r.t. $\mathrm{diag}_b X$

$t = \frac{x'}{x}$



b-principal symbol

${}^b\sigma(P)$ = princ. symbol of kernel at $\mathrm{diag}_{b,X}$

a function on $N^*\mathrm{diag}_b X$
(symbol)

$\cong {}^bT^*X$

$${}^b\sigma(P)(x,y;\lambda,\eta) = \sum_{k+|\alpha|=m} a_{k\alpha}(x,y)\,\lambda^k\eta^\alpha$$

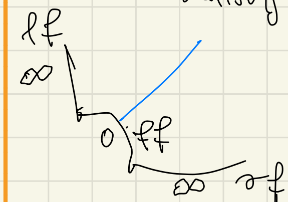
4.1.2 Definition of small b-calculus, elementary mapping properties

Def: Let $m\in\mathbb{R}$, X m.w.bd, ∂X compact,

$\Psi^m_b(X) := \{$ distributions on X^2_b, valued in $|\Omega_b|^{1/2}$,

∘ conormal w.r.t. $\mathrm{diag}_b X$ of order m

∘ vanishing to ∞ order at lf, rf $\}$



(index set $\mathcal{O} = \mathbb{N}_0\times\{0\}$
≙ smooth)

Since $\beta: X^2_b \to X^2$ is diffeo in interior we can interpret these distrib. as operators.

P operator, integral kernel

$$K_P(x,x',y,y')\sqrt{\frac{dt}{x}\frac{dt'}{x}dy\,dy'}$$

P acts on $u(t,y)\sqrt{\frac{dt}{x}dy}$ as

$$(Pu)(x,y) = \boxed{\left(\int K_P(x,x',y,y')\,u(x',y')\,\frac{dt'}{x'}dy'\right)}\sqrt{\frac{dx}{x}dy}$$

Let $\mathcal{U}$ be a nbhd. of ∂X in X

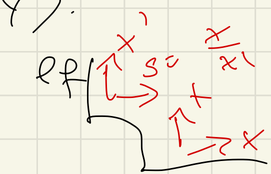
Choose $\mathcal{U} \cong [0,1)_x \times \partial X_y$

If K_P is supported in $\mathcal{U}\times\mathcal{U}$ then the assumptions say:

$$K_P(x,x',y,y') = \kappa_P\left(x,\frac{x'}{x},y,y'\right)$$

where $\kappa_P(x,t,y,y')$ is defined for $t\in(0,\infty)$, $x\in[0,1)$

and
- κ_P conormal wrt $t=1$, $y=y'$
- κ_P vanishes to ∞ order as $t\to 0$ and $t\to\infty$ (uniformly in x, y, y').



[near $t=\infty$:

near lf:

$$K_P = \tilde{\kappa}_P(x',s) \quad (\equiv 0 \text{ as } s\to 0)$$

$$\kappa_P(x,t) = \tilde{\kappa}_P\left(xt,\frac{1}{t}\right)$$]

supported on $x' = xt < 1$.

Prop: $P \in \Psi_b^m(X)$. Then

a) $P: \mathcal{A}^s(X) \to \mathcal{A}^s(X)$

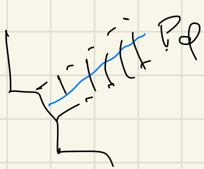
b) $P: \mathcal{A}^{\mathcal{E}}(X) \to \mathcal{A}^{\mathcal{E}}(X)$

$\mathcal{A}^s(X) = \{ u \text{ smooth in } X^\circ;\ Q u = O(x^s)\ \forall Q \in \mathrm{Diff}_b^* \}$

$s \in \mathbb{R}$, $\mathcal{E}$ index set.

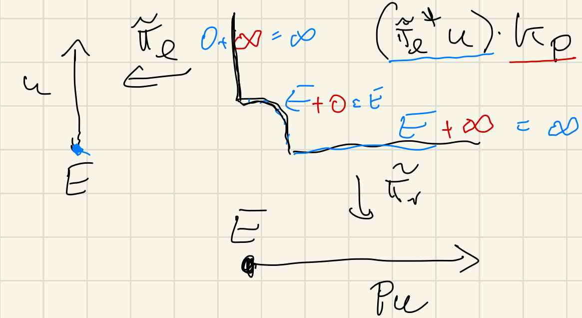
Proof: Write $P = P_\infty + P_d$

where $P_\infty \in \Psi_b^{-\infty}$, K_{P_d} supported near $\mathrm{diag}_{b,X}$



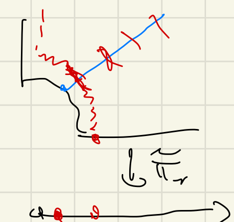
For P_∞ use pull-back and push-forward

thm:



For P_d use PFT for conormal distributions.

Let Z mwc, $Y \subset Z$ interior p-submfd, $\mathcal{E}$ index family for Z, $m \in \mathbb{R}$.



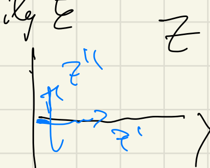
u is conormal of order m w.r.t. Y, phg with index family $\mathcal{E}$

$u \in \mathcal{D}'(Z)$

$:\Leftrightarrow$ · u $\mathcal{E}$-smooth on $Z \setminus Y$

· $u(z) = \int_{\mathbb{R}^l} e^{i\xi z''} a(z',\xi)\, d\xi$ near Y

a phg on $Y \times \overline{\mathbb{R}^l}$ wrt $\mathcal{E}_Y$, $-m$ at $\partial \overline{\mathbb{R}^l}$



PFT in this case:

u phg conormal, $f: Z \to X$

(b-) fibration transversal to Y

$\Rightarrow f_* u$ phg for induced index family.

(end of pf of Prop.)

Special cases:

- $E = \emptyset$: $\dot{C}^\infty(X) = \mathcal{A}^\emptyset(X) = \bigcap \mathcal{A}^s(X)$

  = smooth fcns on X vanishing to ∞ order at ∂X.

  $P: \dot{C}^\infty(X) \to \dot{C}^\infty(X)$

- $E = 0$: $P: C^\infty(X) \to C^\infty(X)$.

4.1.3 b-Symbol, composition, parametrix

Def: ${}^b\sigma_m : \Psi_b^m(X) \to S^{[m]}({}^bT^*X)$

is defined the same way as in Diff_b^m

Thm (Algebra):

$\Psi_b^*(X)$ is a graded algebra and

$\sigma: \Psi_b^*(X) \to S^{[*]}({}^bT^*X)$

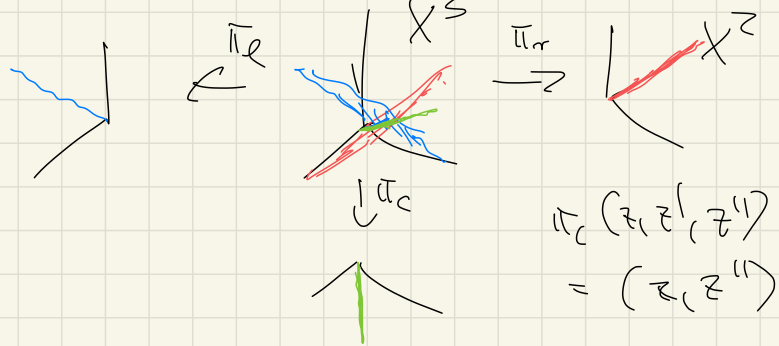
is an algebra homom.:

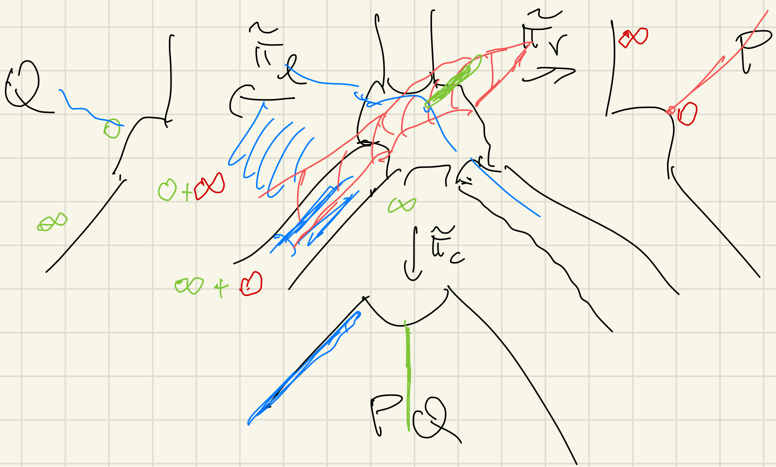
$P \in \Psi_b^m, Q \in \Psi_b^\ell \Rightarrow PQ \in \Psi_b^{m+\ell}$

${}^b\sigma(PQ) = {}^b\sigma(P)\,{}^b\sigma(Q)$

Sketch of proof

$$K_{PQ} = (\pi_c)_* \left(\pi_r^* K_P \cdot \pi_\ell^* K_Q \right)$$





- ∞ at lf, rf: ✓ as above
- conormal singularity at diagonal.
- in interior: same local calculation as for Ψ

This extends uniformly to ff since the geometry is uniform to ff.

qed

<u>Thus</u> (Exact): the sequence

$$0 \to \Psi_b^{m-1}(X) \hookrightarrow \Psi_b^m(X) \xrightarrow{{}^b\sigma_m} S^{[m]}({}^bT^*X) \to 0$$

is exact.
(Clear since true for conormal distrib.)

Thm (AC): $P_j \in \Psi_b^{m-j}$ given

$\Rightarrow \exists P \in \Psi^m : \quad P \sim \sum_{j=0}^{\infty} P_j$.

(as before)

Def: $P \in \Psi_b^m(X)$ is b-elliptic

$:\Leftrightarrow \quad {}^b\sigma(P)$ is invertible on ${}^bT^*X \setminus 0$

Ex: Δ in $\mathbb{R}^2$ in polar coord x, y (radius, angle)

$P = x^2\Delta = (x\partial_x)^2 + \partial_y^2$

${}^b\sigma(P) = -\lambda^2 - \eta^2 \neq 0$ for $(\lambda, \eta) \neq 0$.

$\Rightarrow P$ elliptic.

Note: P is not uniformly elliptic in the usual sense: $\frac{1}{i}\partial_x \rightsquigarrow \xi$, $\frac{1}{i}\partial_y \rightsquigarrow \eta$: $\sigma(P) = -x^2\xi^2 - \eta^2$

at $x=0$: not inv. ($\xi = 1, \eta = 0$).

in b-sense it is uniformly elliptic as $x \to 0$.

Thm: $P \in \Psi_b^m$ b-elliptic

$\Rightarrow \exists Q \in \Psi_b^{-m}$

$PQ = I + R, \quad QP = I + R', \quad R, R' \in \Psi_b^{-\infty}$.

4.1.4 Conormal regularity.

(X compact)

Thm: $P \in \Psi_b^m(X)$ b-elliptic.

Let $u \in \mathcal{D}'(X)$ and assume

$$Pu = f \in \mathcal{A}(X) = \bigcup_s \mathcal{A}^s(X)$$

Then $u \in \mathcal{A}(X)$.

This follows from:

Lemma: $u \in \mathcal{D}'(X)$, $R \in \Psi_b^{-\infty}(X)$

$$\Rightarrow \quad Ru \in \mathcal{A}(X).$$

$$\Big[\quad Pu = f \Rightarrow u + \underbrace{Ru}_{\mathcal{A}} = QPu = \underbrace{Qf}_{\mathcal{A}} \quad\Big]$$

Distributions on mwc:

Def: $\mathcal{D}'(X) := C^{-\infty}(X) =$ dual space of $\dot{C}^\infty(X)$

is called space of extendible distrib.

(or Schwartz type distr.)

[there are also „supported" distr.:
$\dot{C}^{-\infty}(X) =$ dual of $C^\infty(X)$]

Topology on $\dot{C}^\infty(X)$: Fréchet with seminorms $\varphi \in \dot{C}^\infty(X)$

$$q_{N,Q}(\varphi) := \sup_X |x^{-N} Q\varphi|$$

for $N \in \mathbb{N}$, $Q \in \mathrm{Diff}_b^*(X)$.

So

$$u \in C^{-\infty}(X) \iff \exists N, Q's, C$$

$$(*) \quad |\langle u, \varphi \rangle| \le C \sum_{\substack{Q \\ \text{finite}}} q_{N,Q}(\varphi) \quad \forall \varphi.$$

<u>Note</u>: If $X = \overline{\mathbb{R}^n}$ then

$$\dot{C}^\infty(X) = \mathcal{S}(\mathbb{R}^n)$$

so $$C^{-\infty}(X) = \mathcal{S}'(\mathbb{R}^n)$$

<u>Proof of lemma</u>:

Let $u \in C^{-\infty}(X)$, $R \in \Psi_b^{-\infty}(X)$.

<u>Claim</u>: $Ru \in \mathcal{A}^{-N}(X)$

<u>Pf</u>: W.l.o.g. assume K_R supported near $\partial X \times \partial X$. (suppress y-variables)

$$(Ru)(x) = \int K\left(x, \frac{x'}{x}\right) u(x') \frac{dx'}{x'}$$

$$= \langle u, \underbrace{K\left(x, \frac{\cdot}{x}\right)}_{\varphi_x} \rangle_{\frac{dx'}{x'}}$$

$\Rightarrow$

$|Ru(x)| \le$ as in $(*)$

rapid decay as $t \to 0$ or $t \to \infty$

and $$q_{N,Q}(\varphi_x) = \sup \left| (x')^{-N} Q_{x'} \underbrace{\left(k\left(x, \frac{x'}{x}\right)\right)}_{k(x,t)} \right|$$

If $Q = \mathrm{Id}$: $\Rightarrow q(\varphi_x) \approx x^{-N}$ $x \approx x'$

($k(x,t)$ "essentially supported" in $t \in [\frac{1}{C}, C]$)

Q general: $Q \overset{\text{ex}}{=} (x'\partial_{x'})^m k\left(x, \frac{x'}{x}\right)$

$=$ has same prop's as k

+ similar estimates for $A\,Ru$, $A \in \mathrm{Diff}_b^*$. q.e.d.