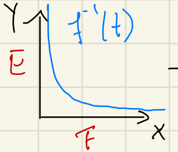
2020-12-03

$f:$  $\longrightarrow$ t (with t marked on the line)

$f(x,y) = x \cdot y$

$$\mu = u \frac{dx}{x}\frac{dy}{y} \Rightarrow f_* \mu = v(t)\frac{dt}{t}$$

$$\boxed{v(t) = \int_0^\infty u\left(x, \frac{t}{x}\right)\frac{dx}{x}}$$

Tbm: $\boxed{u \in \mathcal{A}_0^{E,F}(\mathbb{R}_+^2) \Rightarrow v \in \mathcal{A}_0^{E \bar{\cup} F}(\mathbb{R}_+)}$

If $u \in C_0^\infty(\mathbb{R}_+^2)$ then

$$v(t) \sim \sum_{m=0}^{\infty}\left[\int_0^\infty \frac{1}{x^m} b_m(x)\frac{dx}{x} + \int_0^\infty \frac{1}{y^m} a_m(y)\frac{dy}{y}\right] t^m$$

$$\sim \sum_{m=0}^{\infty} C_{mm}\, t^m \log t \qquad (t \to 0)$$

where

$u \sim \sum_{m=0}^{\infty} a_m(y)\, x^m$ — Taylor as $x \to 0$

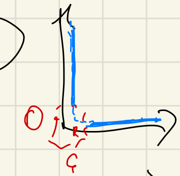
$u \sim \sum_0^\infty b_m(x)\, y^m$ — $y \to 0$

$u \sim \sum^\infty C_{m\ell}\, x^m y^\ell$ — both $x, y \to 0$

Proof: ① $u = 0$ near $(0,0)$

② $u = u_1 \otimes u_2$

③ General case.

① 

Part near x-axis:

$u(x,y) \sim \sum b_m(x)\, y^m \quad (y \to 0)$

uniformly for $x \in [\varepsilon, C]$, $\varepsilon > 0$, $C < \infty$.

$\operatorname{supp} u \subset \{x \le C\}$

$$\Rightarrow u\left(x, \frac{t}{x}\right) \sim \sum b_m(x)\frac{t^m}{x^m} \quad \text{as } t \to 0$$

uniformly for $x \in [\varepsilon, C]$

$$\Rightarrow \int_\varepsilon^C u\left(x, \frac{t}{x}\right)\frac{dx}{x} \sim \sum_m \left(\int_\varepsilon^C \frac{1}{x^m} b_m(x)\frac{dx}{x}\right) t^m$$

$$\underbrace{\qquad}_{= \int_0^\infty \frac{1}{x^m} b_m(x)\frac{dx}{x}}$$

② $u(x,y) = u_1(x) \cdot u_2(y)$.

$$(\mathcal{M}v)(s) = \int_0^\infty v(t)\, t^s \frac{dt}{t} = \int_0^\infty\int_0^\infty u\left(x, \frac{t}{x}\right)\frac{dx}{x}\, t^s \frac{dt}{t}$$

$\left(\frac{t}{x} = y\right)$

$$= \int_0^\infty\int_0^\infty u(x,y)\, x^s y^s \frac{dx}{x}\frac{dy}{y}$$

$$= \int_0^\infty u_1(x)\, x^s \frac{dx}{x} \cdots = (\mathcal{M}u_1)(s) \cdot (\mathcal{M}u_2)(s)$$

$u = u_1 \otimes u_2 \Rightarrow \mathcal{M}v = \mathcal{M}u_1 \cdot \mathcal{M}u_2$

u_1, u_2 phg $\Rightarrow \mathcal{M}u_i$ meromorphic on $\mathbb{C}$

$\Rightarrow \mathcal{M}v \quad \ldots \quad \Rightarrow v$ phg.

(also: bounds as $|\operatorname{Im} s| \to \infty$)

Pole at $s = -m$, $m \in \mathbb{N}_0$.

$\mathcal{M}u_i(s)$: coeff of $(s+m)^{-1}$ is $\frac{1}{m!}\partial_x^m u_i(0)$

coeff of $(s+m)^0$ is $\int_0^\infty \frac{u_i(x)}{x^m} \frac{dx}{x}$

$\Rightarrow \mathcal{M}u_1 \cdot \mathcal{M}u_2$: coeff of $(s+m)^{-2}$ is $\underbrace{\frac{1}{m!}\partial_x^m u_1(0) \cdot \frac{1}{m!}\partial_y^m u_2(0)}_{c_{mm}}$

$= -$ coeff of $t^m \log t$

coeff of $(s+m)^{-1}$: sum of two terms, one involving $\frac{1}{m!}\partial_x^m u_1(0) \cdot \int_0^\infty \frac{u_2(y)}{y^m}\frac{dy}{y}$

+ the other similar $(x \leftrightarrow y)$

qed (2).

(3) $u(x,y) = \underbrace{\sum_{m=0}^{N-1} a_m(y) x^m}_{(2)} + \underbrace{r_N(x,y) x^N}$

$u = u \cdot \chi(x)\chi(y)$

$\rightarrow \underbrace{\sum b_{N,\ell}\, y^\ell x^N}_{(2)} + \underbrace{x^N y^N \cdot c(x,y)}_{\text{Case (1) to order } N.}$

$\rightarrow$ pushforward: $x^N y^N = t^N$

$\rightsquigarrow$ (1) works up to order N incl.

q.e.d.

Obvious generalizations:

- $f(x) = x^\alpha$, $\alpha \in \mathbb{N}$: $f_*\left(u(x)\frac{dx}{x}\right) = u(t^{1/\alpha})\frac{1}{\alpha}\frac{dt}{t}$
  $t = x^\alpha$

$\Rightarrow \mu \in \mathcal{A}^E(\mathbb{R}_+, |\Omega_b|)$ then $f_*\mu \in \mathcal{A}^{\frac{1}{\alpha}E}(\mathbb{R}_+, |\Omega_b|)$

where $cE := \{(cz, k) : (z,k) \in E\}$

for $c > 0$.

- $f(x,y) = x^\alpha y^\beta$. $\quad x^\alpha = \xi$, $y^\beta = \eta$

$= \xi \cdot \eta$

$\Rightarrow$ If $\mu \in \mathcal{A}_0^{E,F}(\mathbb{R}^2_+, |\Omega_b|)$

then $f_*\mu \in \mathcal{A}_0^{\frac{1}{\alpha}E \,\bar\cup\, \frac{1}{\beta}F}(\mathbb{R}_+, |\Omega_b|)$

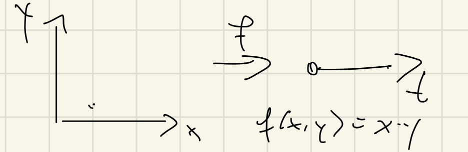
- $f(x_1,\dots,x_n) = x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}$

$\Rightarrow \mu \in \mathcal{A}_0^{(E_1\dots E_n)}(\mathbb{R}^k_+, |\Omega_b|)$

then $f_*\mu \in \mathcal{A}_0^{\frac{1}{\alpha_1}E_1 \,\bar\cup \dots \bar\cup\, \frac{1}{\alpha_\ell}E_\ell}(\mathbb{R}_+, |\Omega_b|)$

$$E \,\bar\cup\, F = E \cup F \cup \{(z, k+\ell+1) : (z,k)\in E,\ (z,\ell)\in F\}$$

Second proof:



- $f_*(x\partial_x) = f_*(y\partial_y) = t\partial_t$
- $\Rightarrow f_*(x\partial_x u) = t\partial_t f_*(u) = f_*(y\partial_y u)$
- bounds: $u = O(x^s y^s) \Rightarrow f_* u = O(t^s \log t)$
  $= O(t^{s-\varepsilon})\ \forall \varepsilon > 0$

$\left|\int_0^\infty u\left(x, \frac{t}{x}\right)\frac{dx}{x}\right| = \left|\int_{t/C}^{C} \dots\right| \le t^s \cdot \int_{t/C}^{C} \frac{dx}{x}$ ✓

$\operatorname{supp} u \subset [0, C]^2$

- conormal regularity:
  $u \in \mathcal{A}^{(s,s)}(\mathbb{R}^2_+) \Rightarrow f_* u \in \mathcal{A}^{s-\varepsilon}(\mathbb{R}_+)\ \forall \varepsilon$.

  Proof: $(t\partial_t)^N f_* u = f_*\left((x\partial_x)^N u\right)$

- polyhomogeneity: $u \in \mathcal{A}^{E,F}$

$\prod_{\substack{(z,k)\in E \\ (w,\ell)\in F}} (x\partial_x - z)(y\partial_y - w)\, u \in \mathcal{A}^{(s,s)}(\mathbb{R}^2_+)$

Apply f_* $\Rightarrow$

$\prod (t\partial_t - z)(t\partial_t - w)\, f_* u \in \mathcal{A}^{s-\varepsilon}(\mathbb{R}_+)\ \forall \varepsilon$.

$z = w$: $k+1 + \ell+1 = k+\ell+2 \sim \log^{k+\ell+1}$ QED

3.3.3 Push-forward theorem

Is it true that $f_*(\text{phg}) = \text{phg}$ for all b-maps $f : X \to Y$?

- interior smoothness ⇝ need f fibration over interior

- need integrability: $X = Y \times Z$, $\downarrow f$, Y

(e.g. $Y = Z = \mathbb{R}_+$)

$$f_*\left(u(y,z)\,\frac{dy}{y}\,\frac{dz}{z}\right) = \left(\int_0^\infty u(y,z)\,\frac{dz}{z}\right)\frac{dy}{y}$$

Need: $F > 0$ ($:\iff \mathrm{Re}\,\tilde{z} > 0 \ \forall (\tilde{z}, k) \in F$)

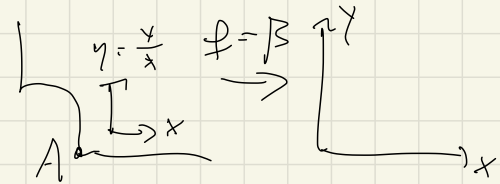
for existence of integral:

$$\int_0^1 z^{\tilde{z}}\,\frac{dz}{z} < \infty \quad \text{iff} \quad \tilde{z} > 0.$$

- If $\dim Y = 1$: no further conditions. (follows from previous case)

- $\dim Y \geq 2$: need more conditions!

example:

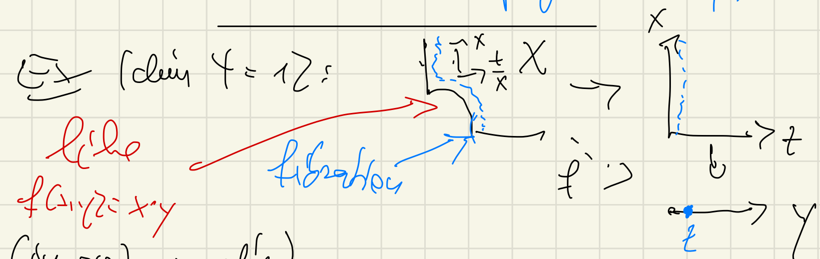


near A:

$$a(x,\eta)\,\frac{dx}{x}\,\frac{d\eta}{\eta} \overset{f_*}{\longmapsto} a\!\left(x, \frac{y}{x}\right)\frac{dx}{x}\,\frac{dy}{y}$$

not phg in (x,y)

Ex ($\dim Y = 1$):

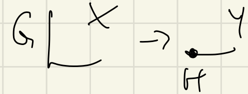


(in proj. coord's)

split up into 2 subproblems ⇝ ✓.

[singular asymptotics lemma]

b-fibrations



Recall b-maps: $f: X \to Y$

bhs	G	H
bdf	ρ_G	ρ'_H

$$f^* \rho'_H = a_H \cdot \prod \rho_G^{e(G,H)}, \quad a_H > 0$$

$e(G,H) \in \mathbb{N}_0$.

Notation: $G \underset{f}{\sim} H \;:\Leftrightarrow\; e(G,H) > 0$

$$G \underset{f}{\sim} H \Leftrightarrow G \subset f^{-1}(H) \Leftrightarrow f(G) \subset H.$$

also: $G \underset{f}{\not\sim} H \Rightarrow f(G^\circ) \cap H = \emptyset$.

This implies $\forall H$: $f^{-1}(H) = \bigcup_{G \underset{f}{\sim} H} G$

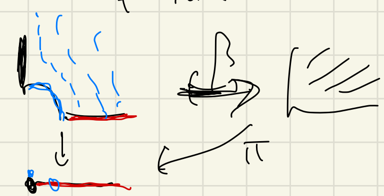
$\forall G$: $f(G) \subset \bigcap_{G \underset{f}{\sim} H} H$

and $f(\mathring{G}) \subset \left(\bigcap_{G \underset{f}{\sim} H} H\right)^\circ$

Lemma: $f: X \to Y$ b-map. Then for each face $F \in \mathcal{M}(X)$ there is a unique face $F' \in \mathcal{M}(Y)$ so that
$$p \in F^\circ \Rightarrow f(p) \in (F')^\circ.$$

Notation: $\bar{f}: \mathcal{M}(X) \to \mathcal{M}(Y)$, $F \mapsto F'$.

ex:



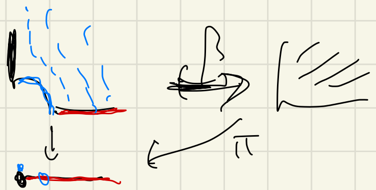
Def: A b-map $X \to Y$ is a <u>b-fibration</u> if it is surjective and for each $F \in \mathcal{M}(X)$:

a) $\operatorname{codim} \bar{f}(F) \leq \operatorname{codim} F$

b) $f|_{\mathring{F}} \to \bar{f}(F)^\circ$ is a fibration.

Note: $X \in \mathcal{M}(X)$

ex:



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Note: $X \in \mathcal{M}(X)$

Rem: a) $\Leftarrow$ a) holds for all $F \in \mathcal{M}_1(X)$.

so a) $\Leftrightarrow$ each bhs G of X then $f(G)$: { bhs of Y / or Y }

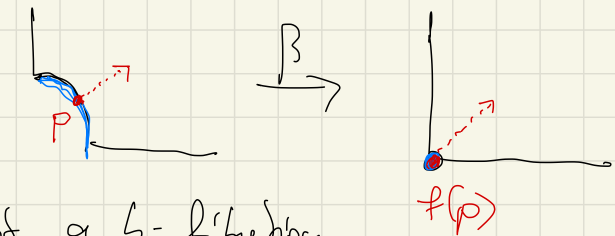
. If f is proper then

← b-Ehresmann theorem

a), b) $\Leftarrow$

- $f_*: {}^b T_p X \to {}^b T_{f(p)} Y$ surjective $\forall p$. (b-submersion)
- $f_*: {}^b N_p F \to {}^b N_{f(p)} F'$ surjective $\forall p \in \mathring{F}$, $\forall F$ (b-normal), $F' = f(F)$

ex:



is not a b-fibration.