b-maps

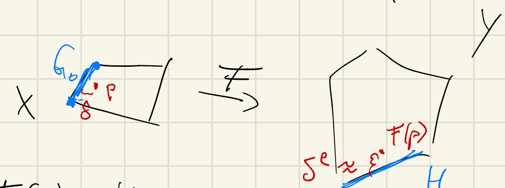
Def: Let X, Y be mwc, $F: X \to Y$ smooth.
Choose bdf's ρ_G for $G \in M_1(X)$, ρ'_H for $H \in M_1(Y)$.
F is a b-map if for each H:

(i) either $F^*\rho'_H = 0$

(ii) or $F^*\rho'_H = a_H \cdot \prod_{G \in M_1(X)} \rho_G^{e(G,H)}$

where $a_H > 0$ smooth, $e(G,H) \in \mathbb{N}_0$ $\forall G, H$.

Geometrically:



(i) $F^*\rho_H = 0 \Leftrightarrow F(X) \subset H$

(ii) Fix G_0, H.

Case a): $e(G_0,H) = 0$: $F(\mathring{G}_0) \cap H = \emptyset$

Case b): $e = e(G_0,H) > 0$: $F^*(\rho'_H) = \rho_{G_0}^e \cdot \tilde{a}$

where $\tilde{a} > 0$ in a nbhd. of G_0°.

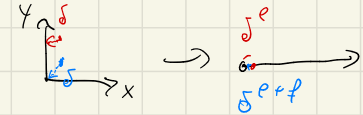
That is: $\rho'_H(F(p)) = \tilde{a}(p) \cdot [\rho_{G_0}(p)]^e \approx [\rho_{G_0}(p)]^e$

So: p has Distance $\sim \delta$ from G_0°

$\Rightarrow F(p)$ - - - - - - $\sim \delta^e$ from H.

Ex: $F(x,y) = x^e y^f$

$F: \mathbb{R}_+^2 \to \mathbb{R}_+$



Note: F b-map $\Rightarrow$ $\forall H \in M_1(Y)$

either: $F^{-1}(H)$ is all of X

or: $F^{-1}(H)$ = union of bh's of X.
(those G having $e(G,H) > 0$)

Def: A b-map is a boundary b-map if $F(X) \subset \partial Y$
otherwise it is an interior b-map.

Lemma: F, G b-maps $\Rightarrow$ $F \circ G$ is b-map.

Rem: Why b-maps? One reason, in ex $\underset{x,y}{\mathbb{R}^2_+} \xrightarrow{F} \underset{t}{\mathbb{R}_+}$

$\log t$, t^z, pull back under F:

b: $F(x,y) = xy$: $\log(xy) = \log x + \log y$, $(xy)^z = x^z y^z$

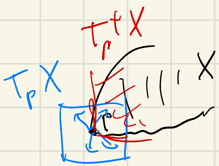
not b: $F(x,y) = x+y$: $\log(x+y) = ?$?

II.4 b-vector fields

Vector fields are central to geometry and analysis:

- first order partial differential operator
  ⇝ generate all linear P.D.O.
- useful tool in geometric constructions

Def: X with mwc; a vector field is a map
$p \mapsto V_p \in T_pX$, smooth in p.



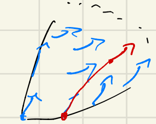
Recall: T_p^+X = inward pointing tangent space

$$= \{ v \in T_pX : \exists \varepsilon > 0\ \exists \gamma : [0,\varepsilon) \to X,\ \gamma(0) = p,\ \dot\gamma(0) = v \}.$$

$\mathcal{V}(X) = \{$all vector fields on $X\}$

$V \in \mathcal{V}(X)$ is inward-pointing $:\Leftrightarrow V_p \in T_p^+X\ \forall p$.

Flows of vector fields:



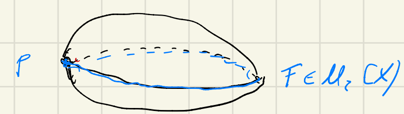
X with mwc, $V \in \mathcal{V}(X)$ inward pointing.
For $K \subset\subset X$ $\exists \varepsilon > 0$ and $\phi : [0,\varepsilon) \times K \to X$
so that for each $p \in K$
$t \mapsto \phi(t,p)$ is an integral curve of V.

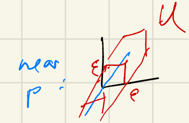
Example of geometric use of vector fields:
Tubular neighborhood theorem: Boundary faces have trivial tubular neighborhoods.

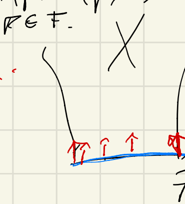
Let X be a compact mwc, $F \in M_k(X)$.
Then $\exists \varepsilon > 0$ and a diffeo

$$\Psi : U \to F \times [0,\varepsilon)^k, \quad \Psi(p) = (p,0) \text{ for } p \in F,$$

for a nbhd. U of F.



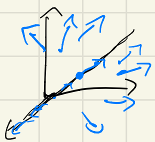




Sketch of proof: Say $k=1$. Choose inward pointing vector field V, not tangential to H, first in local coords, then globally using a partition of unity
← but tang. to all other bhs's.
Then use the flow of V to get Ψ. q.e.d.

Def: A b-vector field is a $V \in \mathcal{V}(X)$ which is tangential to all bhs's of X.

ex.:



$\mathbb{R}^3_2$.

equivalently: V and $-V$ are inward pointing.

Prop: Let X be a compact weak mwc. Then each b-vector field V has a flow, defined for all time $t \in \mathbb{R}$.

Notation: $\mathcal{V}_b(X) = \{b\text{-vector fields}\}$.

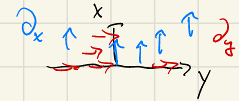
b-vector in coordinates:

Lemma: In any coord. system $(x_1, \dots, x_k, y_1, \dots, y_{n-k})$ a b-vector field has the form

$$V = \sum_{i=1}^{k} a_i \, x_i \partial_{x_i} + \sum_{j=1}^{n-k} b_j \, \partial_{y_j}$$

with a_i, b_j smooth.

Proof:

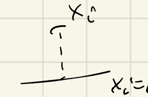


Any vector field has the form $V = \sum A_i \partial_{x_i} + \sum B_j \partial_{y_j}$

A_i, B_j smooth.

V is b-vector field:

at $x_i = 0$ then ∂_{x_i}-component vanishes



$\Rightarrow$ for each i, $A_i = 0$ if $x_i = 0$.

$\underset{\text{Taylor}}{\Rightarrow}$ $A_i = x_i a_i$, a_i smooth.

qed.

Example of singular analysis problem:

Let $M = \{(x,y,z)\in\mathbb{R}^3 : z > \sqrt{x^2+y^2}\}$

Consider the PDE problem

$$\Delta u = f \quad \text{in } M \qquad\qquad \Delta = \partial_x^2+\partial_y^2+\partial_z^2$$
$$u = 0 \quad \text{on } \partial M \setminus \{0\}$$

Questions:

- behavior of $u(p)$ as $p \to 0$.
- Existence, uniqueness
- if yes, understand the solution operator $f \mapsto u$.

Introduce polar coordinates: $p = r\cdot\omega$, $r>0$, $r = |p|$, $\omega \in S^2 = \{\omega\in\mathbb{R}^3 : |\omega| = 1\}$.

In these coord's:

$$\tilde\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta^S$$

where Δ^S = Laplace-Beltrami-Op. on S^2.

$\Delta u = f \iff \tilde\Delta\tilde u = \tilde f$ where $\tilde u(r,\omega) = u(r\omega)$ etc.

$$\tilde\Delta = r^{-2}\left[r^2\partial_r^2 + 2r\partial_r + \Delta^S\right] = r^{-2}P \qquad P = (r\partial_r)^2 + r\partial_r + \Delta^S$$

$$\tilde\Delta\tilde u = \tilde f \iff P\tilde u = r^2\tilde f$$

Separation of variables: $\tilde u(r,\omega) = \sum_j a_j(r)\, u_j(\omega)$

$$-\Delta^S u_j = \lambda_j u_j \quad \text{on } \Omega = M\cap S^2$$
$$u_j = 0 \quad \text{at } \partial\Omega$$

Fact: there are $0 < \lambda_1 \le \lambda_2 \le \dots \to \infty$, $u_1, u_2, \dots$ so that (u_j) are an orthonormal basis of $L^2(\Omega)$.

First, assume $f = 0$ near $r = 0$.

Then: (r near 0) $0 = P\tilde u = \sum_j \left((r\partial_r)^2 + r\partial_r - \lambda_j\right) a_j(r)\cdot u_j(\omega)$

$$\Rightarrow \left[(r\partial_r)^2 + r\partial_r - \lambda_j\right] a_j = 0 \qquad r^{z_j^+}$$

$a_j(r) = r^z$ where $[z^2 + z - \lambda_j] = 0$.

$r\partial_r r^z = z\cdot r^z$

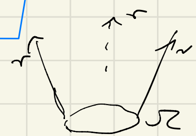
$$\iff z_j^\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda_j}.$$

$$\Rightarrow \boxed{\tilde u(r,\omega) = \sum_{j,\pm} r^{z_j^\pm}\cdot c_j\cdot u_j(\omega).} \qquad z_j^+ > 0, \quad z_j^- < -1.$$

More generally, if e.g. $r^2\tilde f = r^{z_j^+} u_j(\omega)$ then

$$\Rightarrow a_j = r^{z_j^+}\cdot\log r + \dots \qquad (+\, r^{z_j^-})$$

$$\boxed{\tilde u(r,\omega) = \sum_{j,\pm} r^{z_j^\pm} \cdot c_j \cdot \varphi_j(\omega).}$$

Polar coord: M  $\tilde M$ 

p ↔ u (r,ω) ↔ $\tilde u$ $[0,\infty)\times\Omega$ ↔ r

- 1^{st} step: translate problem from M to $\tilde M$. (⇝ blow-up).

(radial) Compactify: $X = [0,\infty) \times \Omega$ is a mwc!

- $\tilde u$ is sum of products $r^z \log r \cdot \varphi_j(\omega)$
  ⇝ polyhomogeneous functions.
- $\tilde u$ can blow up as $r \to 0$ (negative z)
  ⇝ should impose growth conditions on u (and f) as $r \to 0$.
- P is built from $r\partial_r$ and ω-derivatives, i.e. from vector fields tangent to $r = 0$.

$\{r=0\}$ ≙ singularity. ⇝ role of b-vector fields.

dependence of u on f:

$$u(x) = \int_M k(x,y) f(y)\,dy \qquad x \in M$$

(in $\mathbb{R}^3$: $k(x,y) = \frac{c}{|x-y|}$).

Main problems: understand von K. ⇝ b-calculus

k is function (distribution) on $M \times M$.

(⇝ important that $X \times X$ is mwc).

- what happens for more general op's, e.g.

$$\Delta ⇝ \sum a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum b_i(x)\partial_i + c(x)$$

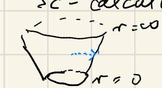
elliptic on $\mathbb{R}^3$, restrict to M.

 – introduce polar coord's.
 – do "approximate sep. of var." ⇝ to find order at $r=0$
 ⇒ improve iteratively.

- what happens as $r \to \infty$?

$r = \frac{1}{s}$ $\Delta = (s^2\partial_s)^2 - s^3\partial_s + s^2 \cdot \underline{\Delta^S}$

⇝ get $s^2\partial_s$, ω-derivatives, $s\cdot\partial_{\omega_i}$ ⇝ different structure at ∞ ⇝ sc-calculus

⇝ consider compactification: $X =$ 

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